

# Nonequilibrium identities of granular vibrating beds

Hisao Hayakawa<sup>1,\*</sup>

<sup>1</sup>*Yukawa Institute for Theoretical Physics,  
Kyoto University, Kyoto 606-8502, Japan*

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## Abstract

We derive the integral fluctuation theorem around a nonequilibrium stationary state for frictionless and soft core granular particles under an external vibration achieved by a balance between an external vibration and inelastic collisions. We also discuss the connection between the integral fluctuation theorem and the generalized Green-Kubo formula.

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\*hisao@yukawa.kyoto-u.ac.jp

## I. INTRODUCTION

One of the most remarkable achievements in recent nonequilibrium statistical mechanics is to demonstrate the existence of some nonequilibrium identities such as the generalized Green-Kubo relation [1, 2], various fluctuation theorems [3–8] and the Jarzynski equality[9] as well as the mutual relationship[10]. These identities are exact and reproduce the conventional Green-Kubo formula, the second law of thermodynamics and Onsager’s reciprocal relation in specific limits. Therefore, these identities are regarded as fundamental relations in nonequilibrium statistical mechanics.

Although it has been believed that these identities are supported by the local time-reversal symmetry or the detailed balance condition, some experiments suggest the existence of fluctuation theorem or related equation even in granular systems which do not have any time reversal symmetry[11–16], though there exists a counter argument [17]. It is remarkable that Puglisi and his coworkers[18–21] clarified that granular fluids do not hold the conventional fluctuation theorem but have only the second type fluctuation theorem by Evans and Searles [22]. As long as the author’s knowledge, however, there is only a paper by Chong et al. which has proven the existence of both the generalized Green-Kubo relation and the integral fluctuation theorem[8] for a granular system under a steady plane shear [12]. They also developed the representation of a nonequilibrium steady-state distribution function [23] and the liquid theory for sheared dense granular systems.[24]. Recently, the present author [25] extended their previous formulation to discuss the nonlinear response theory around a nonequilibrium steady state, and also demonstrate that the conventional fluctuation theorem can be obtained as well as the Jarzynski equality.

Unfortunately, our previous theoretical studies[12, 25] are only valid for sheared cases, but the most of experiments adopt vibrating granular gases[11, 13, 14, 16]. In this paper, thus, we derive the Jarzynski equality for soft core granular gases under vibrations around a nonequilibrium steady state. We also discuss the relationship between the generalized Green-Kubo formula and the integral fluctuation theorem.

The organization of this paper is as follows. In Sec. II, we summarize the general framework of Liouville equation and some identities which are used in this paper. Section III which is the main part of this paper consists of two parts. In the first part (Sec. III A) we discuss the the derivation of the integral fluctuation theorem (IFT). In the second part

(Sec. III B) we also derive a standard fluctuation theorem and the generalized Green-Kubo formula from IFT. In Sec. IV we discuss our results and we give conclusion in section V. In Appendix A, we briefly summarize some operators' identities.

## II. LIOUVILLE EQUATION

Let us consider a system of  $N$  identical soft spherical and smooth dissipative particles. We assume that particles are monodispersed, which are characterized by their diameter  $d$  and the mass  $m$ . The particles are influenced by the gravity with the acceleration constant  $g$  in  $z$ -direction. If we use a box fixed frame, each particle feels the acceleration  $-g + A\omega^2 \cos \omega t$  in  $z$ -direction with the amplitude  $A$  and the angular acceleration  $\omega$ . Moreover, we should introduce a confined potential that particles cannot penetrate the bottom plate.

The basic equation for the statistical mechanics of frictionless granular particles under such a vibrations is the Liouville equation.[1, 26–32] The argument in this section is parallel to that in Ref.[1]. Let  $i\mathcal{L}(t)$  be the total Liouvillian which operates an arbitrary function  $A(\mathbf{\Gamma}(t))$  starting from  $t = 0$  as

$$\frac{dA(\mathbf{\Gamma}(t))}{dt} = U_{\rightarrow}(0, t)i\mathcal{L}(t)A(\mathbf{\Gamma}), \quad A(\mathbf{\Gamma}(t)) = U_{\rightarrow}(0, t)A(\mathbf{\Gamma}), \quad (1)$$

where

$$\begin{aligned} U_{\rightarrow}(0, t) &\equiv T_{\rightarrow} e^{i \int_0^t ds \mathcal{L}(s)} \\ &= \sum_{n=0}^{\infty} \int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{n-1}} ds_n i\mathcal{L}(s_n) \cdots i\mathcal{L}(s_2) i\mathcal{L}(s_1), \end{aligned} \quad (2)$$

and  $\mathbf{\Gamma}(t) = \{\mathbf{r}_i(t), \mathbf{p}_i(t)\}_{i=1}^N$  with the abbreviation  $\mathbf{\Gamma} \equiv \mathbf{\Gamma}(0)$ . We note that there are some trivial relations for  $U_{\rightarrow}(t_0, t)$  such as

$$U_{\rightarrow}(t_0, t) = U_{\rightarrow}(t_0, s)U_{\rightarrow}(s, t); \quad U_{\rightarrow}(t_0, t)\tilde{f}(\mathbf{\Gamma}(t_0)) = \tilde{f}(\mathbf{\Gamma}(t)) \quad (3)$$

for an arbitrary function  $\tilde{f}(\mathbf{\Gamma}(t))$ .

The total Liouvillian consists of three parts, the elastic part, the viscous part and the part from an external vibration. We write  $i\mathcal{L}(t)$  as

$$i\mathcal{L}(t) = i\mathcal{L}^{(\text{el})}(\mathbf{\Gamma}) + i\mathcal{L}^{(\text{vis})}(\mathbf{\Gamma}) + i\mathcal{L}^{\text{ext}}(\mathbf{\Gamma}, t), \quad (4)$$

where  $i\mathcal{L}^{(\text{el})}$  is the elastic collision part,

$$i\mathcal{L}^{(\text{el})}(\Gamma) = \sum_{i=1}^N \frac{\mathbf{p}_i}{m} \cdot \frac{\partial}{\partial \mathbf{r}_i} + \mathbf{F}_i^{(\text{el})} \cdot \frac{\partial}{\partial \mathbf{p}_i}. \quad (5)$$

Here, we assume that the elastic force can be represented by the summation of the pairwise force  $\mathbf{F}_i^{(\text{el})} = \sum_{j \neq i} \mathbf{F}_{ij}^{(\text{el})}$  with

$$\mathbf{F}_{ij}^{(\text{el})} = -\frac{\partial u(r_{ij})}{\partial \mathbf{r}_{ij}} = \Theta(d - r_{ij})f(d - r_{ij})\hat{\mathbf{r}}_{ij}, \quad (6)$$

where we have introduced the pair-wise potential  $u(r_{ij})$ ,  $\mathbf{r}_{ij} \equiv \mathbf{r}_i - \mathbf{r}_j$ ,  $r_{ij} \equiv |\mathbf{r}_{ij}|$ ,  $\hat{\mathbf{r}}_{ij} = \mathbf{r}_{ij}/r_{ij}$ , and the Heviside function  $\Theta(x)$  which satisfies  $\Theta(x) = 1$  for  $x > 0$  and  $\Theta(x) = 0$  for otherwise. The elastic repulsive force  $f(x)$  is proportional to  $x$  for the linear spring model, and to  $x^{3/2}$  for the Hertzian contact model.

Similarly, the viscous Liouvillian  $i\mathcal{L}^{(\text{vis})}$  is the contribution of inelastic collisions:

$$i\mathcal{L}^{(\text{vis})}(\Gamma) = \sum_{i=1}^N \mathbf{F}_i^{(\text{vis})} \cdot \frac{\partial}{\partial \mathbf{p}_i}, \quad (7)$$

where  $\mathbf{F}_i^{(\text{vis})}$  is the viscous force acting on  $i$ -th particle represented by  $\mathbf{F}_i^{(\text{vis})} = \sum_{j \neq i} \mathbf{F}_{ij}^{(\text{vis})}$  with

$$\begin{aligned} \mathbf{F}_{ij}^{(\text{vis})} &= -\hat{\mathbf{r}}_{ij}\Theta(d - r_{ij})\zeta(d - r_{ij})(\mathbf{v}_{ij} \cdot \hat{\mathbf{r}}_{ij}). \\ &= -\hat{\mathbf{r}}_{ij}\mathcal{F}(r_{ij})(\mathbf{v}_{ij} \cdot \hat{\mathbf{r}}_{ij}). \end{aligned} \quad (8)$$

Here we have introduced  $\mathbf{v}_{ij} \equiv \dot{\mathbf{r}}_{ij} = d\mathbf{r}_i/dt$ , and

$$\mathcal{F}(r) \equiv \Theta(d - r)\zeta(d - r) \quad (9)$$

with the viscous function  $\zeta(x)$  which is typically a constant or  $\zeta(x) \propto x^{1/2}$ . The Liouville operator representing the vibration  $i\mathcal{L}^{\text{ext}}(t)$  is given by

$$i\mathcal{L}^{\text{ext}}(\Gamma, t) = \sum_{i=1}^N \mathbf{F}_i^{(\text{ext})}(t) \cdot \frac{\partial}{\partial \mathbf{p}_i}, \quad (10)$$

where the vibrating force is given by

$$\mathbf{F}_i^{(\text{ext})}(t) = \hat{z} \left\{ m(-g + A\omega^2 \cos \omega t) - \frac{\partial V_{\text{ext}}(z_i)}{\partial z_i} \right\} = \hat{z} F_i^{(\text{ext})}(t) \quad (11)$$

in a box fixed frame, where  $\hat{z}$  is the unit vector in  $z$  direction, and  $V_{\text{ext}}(z)$  represents a confined potential in a box such as

$$V_{\text{ext}}(z) = V_0 \exp[-z/\xi] \quad (12)$$

to prevent grains from penetrating the bottom plate of the container.

It should be noted that the Liouvillian is not self-adjoint, because of the violation of time-reversal symmetry for each collision. The adjoint Liouvillian is defined through the equation of the phase function or the  $N$ -body distribution function  $\rho(\mathbf{\Gamma}, t)$

$$\rho(\mathbf{\Gamma}, t) = \tilde{U}_{\leftarrow}(t, 0)\rho(\mathbf{\Gamma}, 0), \quad \frac{\partial \rho(\mathbf{\Gamma}, t)}{\partial t} = -i\mathcal{L}^\dagger(t)\rho(\mathbf{\Gamma}, t), \quad (13)$$

where

$$\begin{aligned} \tilde{U}_{\leftarrow}(t, 0) &= T_{\leftarrow} e^{-i \int_0^t ds \mathcal{L}^\dagger(s)} \\ &\equiv \sum_{n=0}^0 (-)^n \int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{n-1}} ds_n i\mathcal{L}^\dagger(s_1) i\mathcal{L}^\dagger(s_2) \cdots i\mathcal{L}^\dagger(s_n). \end{aligned} \quad (14)$$

The adjoint Liouvillian satisfies

$$i\mathcal{L}^\dagger(\mathbf{\Gamma}, t) = i\mathcal{L}(\mathbf{\Gamma}, t) + \Lambda(\mathbf{\Gamma}), \quad (15)$$

where

$$\Lambda(\mathbf{\Gamma}) \equiv \frac{\partial}{\partial \mathbf{\Gamma}} \cdot \dot{\mathbf{\Gamma}}(\mathbf{\Gamma}) \quad (16)$$

is the phase volume contraction. We note that  $\Lambda(\mathbf{\Gamma})$  in our system does not depend on  $t$  explicitly, which can be written as

$$\Lambda(\mathbf{\Gamma}) = \sum_i \frac{\partial}{\partial \mathbf{p}_i} \cdot \mathbf{F}_i^{(\text{vis})} = -\frac{1}{m} \sum_i \sum_{j \neq i} \mathcal{F}(r_{ij}) \quad (17)$$

for  $t \geq 0$ . The phase volume contraction  $\Lambda(\mathbf{\Gamma})$  is directly related to the change of Jacobian

$$\left| \frac{\partial \mathbf{\Gamma}(t)}{\partial \mathbf{\Gamma}} \right| = \exp \left[ \int_0^t d\tau \Lambda(\mathbf{\Gamma}(\tau)) \right], \quad (18)$$

where  $\Lambda(\mathbf{\Gamma}(t)) = U_{\rightarrow}(0, t)\Lambda(\mathbf{\Gamma})U_{\leftarrow}(t, 0)$ . Note that the time evolution of an arbitrary physical function  $A(\mathbf{\Gamma}(t))$  is given by  $A(\mathbf{\Gamma}(t)) = U_{\rightarrow}(0, t)A(\mathbf{\Gamma})U_{\leftarrow}(t, 0)$ , where we have introduced  $U_{\leftarrow}(t, 0) \equiv T_{\leftarrow} \exp[-i \int_0^t ds \mathcal{L}(s)]$ .

The average of a physical quantity is defined as

$$\langle A(\mathbf{\Gamma}(t)) \rangle \equiv \int d\mathbf{\Gamma} \rho(\mathbf{\Gamma}, 0) A(\mathbf{\Gamma}(t)) = \int d\mathbf{\Gamma} A(\mathbf{\Gamma}) \rho(\mathbf{\Gamma}, t). \quad (19)$$

From Eqs. (1), (13) and (19) we obtain the relations

$$\int d\mathbf{\Gamma} \rho(\mathbf{\Gamma}) U_{\rightarrow}(0, t) A(\mathbf{\Gamma}) = \int d\mathbf{\Gamma} A(\mathbf{\Gamma}) \tilde{U}_{\leftarrow}(t, 0) \rho(\mathbf{\Gamma}, 0) \quad (20)$$

and

$$\int d\mathbf{\Gamma} \rho(\mathbf{\Gamma}) i\mathcal{L}(t) A(\mathbf{\Gamma}) = - \int d\mathbf{\Gamma} A(\mathbf{\Gamma}) i\mathcal{L}^{\dagger}(t) \rho(\mathbf{\Gamma}, 0). \quad (21)$$

In the last part of this section, let us introduce a stationary distribution to characterize the quasi-periodic motion of granular particles under the periodic vibration. In the stationary process, the initial distribution function  $\rho(\mathbf{\Gamma}, 0)$  may have the form

$$\rho(\mathbf{\Gamma}, 0) = \rho_{\text{ini}}(\mathbf{\Gamma}) \equiv \frac{e^{-I_0(\mathbf{\Gamma})}}{\mathcal{Z}}, \quad (22)$$

where  $I_0(\mathbf{\Gamma}) \equiv I(\mathbf{\Gamma}, t = 2n\pi/\omega)$  with an arbitrary integer  $n$ . and  $\mathcal{Z} \equiv \int d\mathbf{\Gamma} e^{-I_0(\mathbf{\Gamma})}$ . Note that  $I_0(\mathbf{\Gamma})$  is an arbitrary function of  $\mathbf{\Gamma}$ , and thus, this choice is quite general for the argument. In the stationary process, we assume that an average of an arbitray function  $A(\mathbf{\Gamma}(t))$  satisfies the perodic condition

$$\left\langle A(\mathbf{\Gamma} \left( t + \frac{2n\pi}{\omega} \right) t \right\rangle = \langle A(\mathbf{\Gamma}(t)) \rangle \quad (23)$$

for any nonzero integer  $n$ . This assumption is reasonable because of the periodicity of the Liouvillian:

$$i\mathcal{L} \left( t + \frac{2n\pi}{\omega} \right) = i\mathcal{L}(t) \quad (24)$$

for an arbitrary integer  $n$ .

### III. FLUCTUATION THEOREM AND GREEN-KUBO FORMULA

This section consists of two parts. The first part is dedicated to the derivation of the integral fluctuation theorem (IFT). The second part discusses the derivations of both the standard fluctuation theorem and the generalized Green-Kubo formula.

#### A. Integral fluctuation theorem

The integral fluctuation theorem (IFT) is a kind of the fluctuation theorem, which is related to Jarzynski equality.[9] The relation between Jarzynski equality and the fluctuation theorem has been investigated extensively.[8, 10] Although IFT is an important identity for

the stationary sheared granular systems[12], and this equality plays a fundamental role even in granular systems under a vibration.

To demonstrate the existence of the IFT, we consider a system characterized by the following time-dependent Hamiltonian :

$$H_0(\mathbf{\Gamma}(t)) = \sum_i \frac{\mathbf{p}_i(t)^2}{2m} + \frac{1}{2} \sum_{i,j \neq i} u(r_{ij}(t)). \quad (25)$$

We also assume that the initial condition satisfies the canonical distribution

$$\rho_{\text{eq}}(\mathbf{\Gamma}) = \frac{e^{-\beta H_0(\mathbf{\Gamma})}}{Z}, \quad (26)$$

where  $\beta$  is the inverse temperature and  $Z(\beta) \equiv \int d\mathbf{\Gamma} e^{-\beta H_0(\mathbf{\Gamma})}$ .

In this case, it is easy to verify the conservation of the normalization factor. i.e.  $Z(\beta) = \int d\mathbf{\Gamma} e^{-\beta H_0(\mathbf{\Gamma})} = \int d\mathbf{\Gamma}(t) e^{-\beta H_0(\mathbf{\Gamma}(t))}$ . Because the time derivative of  $H_0(\mathbf{\Gamma}(t))$  is given by

$$\dot{H}_0(\mathbf{\Gamma}(t)) = \sum_i v_{i,z}(t) F_i^{(\text{ext})}(t) - 2\mathcal{R}(\mathbf{\Gamma}(t)) \quad (27)$$

with  $\dot{H}_0 \equiv dH_0/dt$ ,  $\mathbf{v}_i(t) \equiv d\mathbf{r}_i(t)/dt$ ,

$$\mathcal{R}(\mathbf{\Gamma}) \equiv -\frac{1}{4} \sum_{i,j} \mathbf{v}_{ij} \cdot \mathbf{F}_{ij}^{(\text{vis})} = \frac{1}{4} \sum_{i,j} \mathcal{F}(r_{ij}) (\mathbf{v}_{ij} \cdot \hat{\mathbf{r}}_{ij})^2, \quad (28)$$

and  $\mathbf{v}_{ij} \equiv \mathbf{v}_i - \mathbf{v}_j$ , the conservation of  $Z(\beta)$  satisfies

$$\begin{aligned} 1 &= \int d\mathbf{\Gamma}(t) \frac{e^{-\beta H_0(\mathbf{\Gamma}(t))}}{Z(\beta)} \\ &= \int d\mathbf{\Gamma} \left| \frac{\partial \mathbf{\Gamma}(t)}{\partial \mathbf{\Gamma}} \right| \frac{e^{-\beta H_0(\mathbf{\Gamma})}}{Z(\beta)} \exp \left[ \beta \int_0^t d\tau \left\{ \sum_i v_{i,z}(\tau) F_i^{(\text{ext})}(\tau) - 2\mathcal{R}(\mathbf{\Gamma}(\tau)) \right\} \right] \\ &= \int d\mathbf{\Gamma} \frac{e^{-\beta H_0(\mathbf{\Gamma})}}{Z(\beta)} \exp \left[ - \int_0^t d\tau \Omega_{\text{eq}}(\mathbf{\Gamma}(\tau)) \right] \\ &= \left\langle \exp \left[ - \int_0^t d\tau \Omega_{\text{eq}}(\mathbf{\Gamma}(\tau)) \right] \right\rangle_{\text{eq}}, \end{aligned} \quad (29)$$

where we have used Eq.(18) for the third equality, and introduced

$$\Omega_{\text{eq}}(\mathbf{\Gamma}(t)) \equiv \beta \sum_i v_{i,z}(t) F_i^{(\text{ext})}(t) - 2\beta \mathcal{R}(\mathbf{\Gamma}(t)) - \Lambda(\mathbf{\Gamma}(t)). \quad (30)$$

Equation (29) associated with Eq.(30) is the IFT for granular fluids under the vibration.

It is a characteristic feature for dissipative systems that the phase volume contraction  $\Lambda(\mathbf{\Gamma})$  is involved in  $\Omega_{\text{eq}}(\mathbf{\Gamma}(t))$ . Thus, the right hand side of the Jarzynski equality (29) for dissipative cases cannot be represented by the work done by the external force.

The IFT (29) is directly reduced to an inequality

$$\int_0^t d\tau \langle \Omega_{\text{eq}}(\mathbf{\Gamma}(\tau)) \rangle \geq 0 \quad (31)$$

with the aid of Jenssen's inequality.

Now, let us extend the IFT to the case of starting from an arbitrary distribution  $\rho_{\text{ini}}(\mathbf{\Gamma})$ . In this case, the IFT can be rewritten as

$$\begin{aligned} 1 &= \int d\mathbf{\Gamma}(t) \frac{e^{-I_0(\mathbf{\Gamma}(t))}}{\mathcal{Z}} \\ &= \int d\mathbf{\Gamma} \left| \frac{\partial \mathbf{\Gamma}(t)}{\partial \mathbf{\Gamma}} \right| \frac{e^{-I_0(\mathbf{\Gamma})}}{\mathcal{Z}} \exp \left[ - \int_0^t d\tau \dot{I}_0(\mathbf{\Gamma}(\tau)) \right] \\ &= \int d\mathbf{\Gamma} \frac{e^{-I_0(\mathbf{\Gamma})}}{\mathcal{Z}} \exp \left[ - \int_0^t d\tau \Omega(\mathbf{\Gamma}(\tau)) \right] \\ &= \left\langle \exp \left[ - \int_0^t d\tau \Omega(\mathbf{\Gamma}(\tau)) \right] \right\rangle, \end{aligned} \quad (32)$$

where  $\Omega(\mathbf{\Gamma}(t)) = \dot{I}_0(\mathbf{\Gamma}(t)) - \Lambda(\mathbf{\Gamma}(t))$ , and  $\mathcal{Z} \equiv \int d\mathbf{\Gamma} e^{-I_0(\mathbf{\Gamma})} = \int d\mathbf{\Gamma}(t) e^{-I_0(\mathbf{\Gamma}(t))}$ . Equation (32) is also reduced to

$$\int_0^t d\tau \langle \Omega(\mathbf{\Gamma}(\tau), \tau) \rangle \geq 0. \quad (33)$$

## B. The standard Fluctuation Theorem and Generalized Green-Kubo formula

The direct consequences of Eq. (32) are two important relations, the conventional fluctuation theorem and the generalized Green-Kubo formula, from Eq.(32). Here, let us illustrate how to derive these two relations.

It is straightforward to derive the standard fluctuation theorem from IFT [1], if we assume the time reversal symmetry in  $\rho_{\text{ini}}(\mathbf{\Gamma})$  as

$$\rho_{\text{ini}}(\mathbf{\Gamma}^T) = \rho_{\text{ini}}(\mathbf{\Gamma}), \quad (34)$$

where  $\mathbf{\Gamma}^T$  represents the time reversal operation of  $\mathbf{\Gamma}$ , i.e.  $\mathbf{\Gamma}^T = \{\mathbf{r}_i, -\mathbf{p}_i\}_{i=1}^N$ . Let us derive the standard fluctuation theorem from the integral fluctuation theorem (32). Equation (32) can be rewritten as

$$\int d\mathbf{\Gamma} \rho_{\text{ini}}(\mathbf{\Gamma}) e^{-t\overline{\Omega}_t} = \int d\mathbf{\Gamma}^T \rho_{\text{ini}}(\mathbf{\Gamma}^T), \quad (35)$$

where we have introduced  $\overline{\Omega}_t \equiv \frac{1}{t} \int_0^t d\tau \Omega(\mathbf{\Gamma}(\tau))$  for  $t = 2n\pi/\omega$  for an arbitrary positive integer  $n$  and used the fact that the normalization is unchanged for  $\int d\mathbf{\Gamma}^T \rho_{\text{ini}}(\mathbf{\Gamma}^T) = 1$ . With the



aid of Eq.(34) and  $d\mathbf{\Gamma}^T = d\mathbf{\Gamma}|\partial\mathbf{\Gamma}^T/\partial\mathbf{\Gamma}|$ , Eq. (35) means that the Jacobian  $|\partial\mathbf{\Gamma}^T/\partial\mathbf{\Gamma}|$  is given by  $\exp[-t\bar{\Omega}_t]$ . Therefore, we can write the probability of  $\bar{\tilde{\Omega}}_t = A$  for  $\bar{\tilde{\Omega}}_t \equiv \frac{1}{t} \int_0^t d\tau \tilde{\Omega}(\mathbf{\Gamma}(\tau))$  and  $\tilde{\Omega}(\mathbf{\Gamma}) \equiv \Omega(\mathbf{\Gamma}^T)$  as

$$\begin{aligned} \text{Prob}(\bar{\tilde{\Omega}}_t = A) &= \int d\mathbf{\Gamma}^T \rho_{\text{ini}}(\mathbf{\Gamma}^T) \delta(\bar{\tilde{\Omega}}_t(\mathbf{\Gamma}^T) - A) \\ &= \int d\mathbf{\Gamma} \rho_{\text{ini}}(\mathbf{\Gamma}) e^{-t\bar{\Omega}_t} \delta(\bar{\Omega}_t(\mathbf{\Gamma}) - A) \\ &= e^{-At} \int d\mathbf{\Gamma} \rho_{\text{ini}}(\mathbf{\Gamma}) \delta(\bar{\Omega}_t(\mathbf{\Gamma}) - A) \\ &= e^{-At} \text{Prob}(\bar{\Omega}_t = A), \end{aligned} \quad (36)$$

where we have used  $\tilde{\Omega}(\mathbf{\Gamma}^T) = \Omega(\{\mathbf{\Gamma}^T\}^T) = \Omega(\mathbf{\Gamma})$  in the second line. Note that our fluctuation theorem

$$\text{Prob}(\bar{\tilde{\Omega}}_t = A) = e^{-At} \text{Prob}(\bar{\Omega}_t = A) \quad (37)$$

differs from the standard form, because our system does not have the time reversal symmetry.

Next, let us derive the generalized Green-Kubo formula from Eq.(32) following the argument in Ref.[12]. Equation (32) leads to the relation  $d\mathbf{\Gamma} \rho_{\text{ini}}(\mathbf{\Gamma}) e^{-\int_0^t d\tau \Omega(\mathbf{\Gamma}(\tau))} = d\mathbf{\Gamma}(t) \rho_{\text{ini}}(\mathbf{\Gamma}(t))$  in a co-moving frame. Multiplying  $e^{\int_0^t d\tau \Omega(\mathbf{\Gamma}(\tau))}$  for both side, we obtain

$$\begin{aligned} d\mathbf{\Gamma} \rho_{\text{ini}}(\mathbf{\Gamma}) &= d\mathbf{\Gamma}(t) \rho_{\text{ini}}(\mathbf{\Gamma}(t)) \exp \left[ \int_0^t d\tau \Omega(\mathbf{\Gamma}(\tau)) \right] \\ &= d\tilde{\mathbf{\Gamma}} \rho_{\text{ini}}(\tilde{\mathbf{\Gamma}}) \exp \left[ \int_{-t}^0 d\tau \Omega(\tilde{\mathbf{\Gamma}}(\tau)) \right] \\ &= d\tilde{\mathbf{\Gamma}} \rho_{\text{ini}}(\tilde{\mathbf{\Gamma}}) \exp \left[ \int_0^t d\tau \Omega(\tilde{\mathbf{\Gamma}}(-\tau)) \right], \end{aligned} \quad (38)$$

where we have introduced  $\tilde{\mathbf{\Gamma}} \equiv \mathbf{\Gamma}(t)$  and used  $\tilde{\mathbf{\Gamma}}(\tau) = \mathbf{\Gamma}(t + \tau)$ . Therefore, we reach the conservation of the probability:

$$\langle e^{\int_0^t d\tau \Omega(\mathbf{\Gamma}(-\tau))} \rangle = 1. \quad (39)$$

From Eq.(39), we obtain the time evolution of the distribution function:

$$\rho(\mathbf{\Gamma}, t) = e^{\int_0^t d\tau \Omega(\mathbf{\Gamma}(-\tau))} \rho_{\text{ini}}(\mathbf{\Gamma}). \quad (40)$$

The generalized Green-Kubo formula is the direct consequence of the IFT. The differen-

tiation of Eq.(19) with the help of Eq.(40), we obtain

$$\begin{aligned}
\frac{d}{dt}\langle A(\mathbf{\Gamma}(t)) \rangle &= \int d\mathbf{\Gamma} A(\mathbf{\Gamma}) \Omega(\mathbf{\Gamma}(-t)) \rho(\mathbf{\Gamma}, t) = \int d\mathbf{\Gamma} U_{\leftarrow}(t, 0) \{A(\mathbf{\Gamma}(t)) \Omega(\mathbf{\Gamma})\} \rho(\mathbf{\Gamma}, t) \\
&= \int d\mathbf{\Gamma} A(\mathbf{\Gamma}(t)) \Omega(\mathbf{\Gamma}) \tilde{U}_{\rightarrow}(0, t) \rho(\mathbf{\Gamma}, t) = \int d\mathbf{\Gamma} A(\mathbf{\Gamma}(t)) \Omega(\mathbf{\Gamma}) e^{\int_0^t d\tau \Lambda(\mathbf{\Gamma}(\tau))} \rho(\mathbf{\Gamma}(t), t) \\
&= \int d\mathbf{\Gamma} A(\mathbf{\Gamma}(t)) \Omega(\mathbf{\Gamma}) \frac{e^{-I_0(\mathbf{\Gamma})}}{\mathcal{Z}} = \langle A(\mathbf{\Gamma}(t)) \Omega(\mathbf{\Gamma}) \rangle.
\end{aligned} \tag{41}$$

This equation can be integrated over  $t$  as

$$\langle A(\mathbf{\Gamma}(t)) \rangle = \langle A(\mathbf{\Gamma}) \rangle + \int_0^t ds \langle A(\mathbf{\Gamma}(s)) \Omega(\mathbf{\Gamma}) \rangle. \tag{42}$$

This result depends on  $\rho_{\text{ini}}(\mathbf{\Gamma})$ . Therefore, the formal response theory can be written as

$$\langle \delta A(\mathbf{\Gamma}) \rangle = \int_0^\infty dt \langle A(\mathbf{\Gamma}(t)) \Omega(\mathbf{\Gamma}) \rangle, \tag{43}$$

where  $\delta A(\mathbf{\Gamma}) \equiv \lim_{t \rightarrow \infty} A(\mathbf{\Gamma}(t)) - A(\mathbf{\Gamma})$ .

## IV. DISCUSSION

In this paper, we obtain exact nonequilibrium relations. To verify the validity, we may need numerical simulations. It should be noted that the verification of the generalized Green-Kubo formula is not difficult by the direct simulation, but the confirmation of the integral fluctuation theorem by simulations is not easy because of the limitation of numerical accuracy. Indeed, the normalization condition (39) could not be achieved in the simulation even for thermostat systems.[34] We should stress that the generalized Green-Kubo formula can be used for dense granular systems above the jamming transition.

Even when we confirm the validity of the nonequilibrium relations such as the generalized Green-Kubo formula, it is not easy to calculate the correlation function Eq.(42) or Eq.(43). One of possible methods is to use the mode-coupling theory (MCT). It is helpful to apply MCT for granular liquids to characterize theology near the jamming transition.[29, 36–38] It is notable that Ref.[31] develops a linear response theory for a sheared thermostat system around a nonequilibrium steady state. The application of this method will be discussed elsewhere.

## V. SUMMARY

We have developed some exact relations for frictionless granular fluids under vibrations. We derived the integral fluctuation theorem around a nonequilibrium steady state. We also derived the formal expression of the distribution function. We finally obtained the generalized Green-Kubo formula around a nonequilibrium steady state.

In this paper, we focus on the detailed analytic calculation on the granular fluids under the vibration. The systematic check in terms of the simulations will be reported elsewhere.

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### Appendix A: Some operators’ identities

Let us consider the time evolution of  $\mathbf{\Gamma}(t; t_0)$  defined by

$$\mathbf{\Gamma}(t; t_0) = U_{\rightarrow}(t_0, t)\mathbf{\Gamma}(t_0), \quad (\text{A1})$$

where we have explicitly written the initial time  $t_0$ .

By using  $U_{\rightarrow}(t_0, t)$  and  $U_{\leftarrow}(t, t_0)$  it is notable that there is an important relation for  $U_{\rightarrow}(t_0, t)$ :

$$A(\mathbf{\Gamma}(t)) \equiv U_{\rightarrow}(t_0, t)A(\mathbf{\Gamma}(t_0))U_{\leftarrow}(t, t_0) = U_{\rightarrow}(t_0, t)A(\mathbf{\Gamma}(t_0)). \quad (\text{A2})$$

The proof of (A2) is straightforward. The right hand side of Eq.(A2) can be rewritten as

$$\begin{aligned} U_{\rightarrow}(t_0, t)A(\mathbf{\Gamma}(t_0)) &= U_{\rightarrow}(t_0, t)A(\mathbf{\Gamma}(t_0))U_{\leftarrow}(t, t_0)U_{\rightarrow}(t, t_0)1 = U_{\rightarrow}(t_0, t)A(\mathbf{\Gamma}(t_0))U_{\leftarrow}(t, t_0)1 \\ &= U_{\rightarrow}(t_0, t)A(\mathbf{\Gamma}(t_0))U_{\leftarrow}(t, t_0), \end{aligned} \quad (\text{A3})$$

where we have used  $U_{\rightarrow}(t_0, t)1 = 1$  for a constant 1. When we use Eq.(A2), we readily obtain

$$U_{\rightarrow}(t_0, t)A(\mathbf{\Gamma}(t_0))B(\mathbf{\Gamma}(t_0)) = A(\mathbf{\Gamma}(t; t_0)) \cdot B(\mathbf{\Gamma}(t; t_0)). \quad (\text{A4})$$

Indeed, the left hand side of this equation can be rewritten as

$$\begin{aligned} U_{\rightarrow}(t_0, t)A(\mathbf{\Gamma}(t_0))B(\mathbf{\Gamma}(t_0)) &= U_{\rightarrow}(t_0, t)A(\mathbf{\Gamma}(t_0))U_{\leftarrow}(t, t_0)U_{\rightarrow}(t_0, t)B(\mathbf{\Gamma}(t_0))U_{\leftarrow}(t, t_0) \\ &= A(\mathbf{\Gamma}(t; t_0)) \cdot B(\mathbf{\Gamma}(t; t_0)), \end{aligned} \tag{A5}$$

which is the end of the proof of Eq.(A4).

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